On Galileo's Tallest Column

Mariano Vázquez Espí* Jaime Cervera Bravo† Carlos Olmedo Rojas‡

Madrid, March 31, 2015.

Abstract

The height at which an unloaded column will fail under its own weight was calculated for first time by Galileo for cylindrical columns. Galileo questioned himself if there exists a shape function for the cross-section of the column with which the latter can attains a greater height than the cylindrical column. The problem is not solved since then, although the definition of the so named "constant maximum strength" solids seems to give an affirmative answer to Galileo's question, in the form of shapes than can attains infinite height, even when loaded with a useful load at the top. The main contribution of this work is to show that Galileo's problem is (i) an important problem for structural design theory of buildings and other structures, (ii) not solved by the time being in any sense and (iii) a interesting problem for mathematicians involved in related but very different problems (as Euler's tallest column). A contemporary formulation of the problem is included as a result of a research on the subject.

Keywords: structural design, insurmountable size, structural efficiency, structural scope, optimal shapes.

1 Introduction

In 1638, in his *Discorsi e Dimostrazioni Matematiche*[12], Galileo postulated the existence of the tallest column, i.e., a cylindrical column, such that it attains the maximum height once the area of its cross-section and the strength of its material are prescribed. Therefore, Galileo's tallest column is in the limit of resistance only bearing its own weight. The rationale of his proof gave raise to the Square-cube law, a mathematical principle that considers the relationship between the flow through the surface of a volume and the stock into the latter: in the mechanical case, for example, the stress with the weight. This principle has been very useful and applied in a variety of scientific fields, mainly in biology [15, 18, 24].

In the following century, Euler [11] pointed out a very different problem, i.e., to find the shape of a stable column, axially symmetric respect to the vertical axis, such that it attains the maximum height once its volume, specific weight and Young's module are given, buckling due to the action of a load bear at its top.

In our view, both problems are not yet completely solved nowadays, although Galileo's problem have received much less attention than Euler's. Furthermore, we think that Galileo's problem is more meaningful for a theory of design of structures subjected to small limit on strains and displacements, as buildings and other structures in civil engineering [1, 3].

Hereafter, we consider the solid continuum with the following standard assumptions:

1. The scope of the analysis is the classical theory of Elasticity
2. The process of deformation is isothermal and quasi-static: heat or kinetic energy are not taken in consideration into energy balancing.
3. We are only interested on solutions where strains and displacements will be very small, hence equilibrium and compatibility equations must hold in undeformed body.

Section 2 outlines important aspects of Galileo's problem, comparing it to Euler's, and enlightening its importance and profound meaning for a theory of structural design. Our main working hypothesis is formulated there. Section 3 deals with some clues that support our working hypothesis, i.e., that a insurmountable size exists for a fairly large set of structural problems, as it is the case of cylindrical columns of Galileo's first insight on the tallest column. This section covers the main aim of this paper: to attract mathematicians to work in Galileo's problem, because we are architects and our mathematical

*Grupo de Investigación en Arquitectura, Urbanismo y Sostenibilidad (GIAU+S UPM). email: `mariano.vazquez.espi@upm.es`

†ETS. de Arquitectura de Madrid (UPM) email: `jaime.cervera@upm.es`

‡ETS. de Arquitectura de Madrid (UPM) email: `olmedo.c@gmail.com`

knowledge is, to say the least, limited, but to solve the problem is a key point to continue the development of structural design theory. Finally, Section 4 is devoted to formulate the problem formally in contemporary terms.

2 Galileo's problem on the tallest column

Proposition VII

Among heavy prisms and cylinders of similar figure, there is one and only one which under the stress of its own weight lies just on the limit between breaking and not breaking: so that every larger one is unable to carry the load of its own weight and breaks; while every smaller one is able to withstand some additional force tending to break it.

Galileo, 1638

Consider a cylindrical column of height L and diameter d subjected to the action of its own weight, of a lineal elastic material defined by Young's Modulus $\mathbf{E}$, allowable compressive stress $\mathbf{f}$ and specific weight $\boldsymbol{\rho}$.

Such a column will be unsafe in simple compression if the applied load $\boldsymbol{\rho} LA$ exceeds the column strength $A\mathbf{f}$, where A is the area of the cross-section —if the base is a circle, $A = \frac{1}{4}\pi d^2$— or, which is the same, if the applied stress $\boldsymbol{\rho} L$ exceeds $\mathbf{f}$. This fact means that the height of such a column may not be greater that a characteristic length of the material, $\mathbf{f} \div \boldsymbol{\rho}$. We name this length "structural scope" of the material, $\mathcal{A} = \mathbf{f} \div \boldsymbol{\rho}$. And we name "structural scope" $\mathcal{L}$ of cylindrical columns to the maximum height of safe columns. In this simple case, $\mathcal{L}$ is numerically equal to the material scope $\mathcal{A}$, but generally $\mathcal{L}$ is related with $\mathcal{A}$ but not equal to [1]. Therefore the first conclusion of Galileo can be expressed as:

$$L \leq \mathcal{L} = \frac{\mathbf{f}}{\boldsymbol{\rho}} = \mathcal{A} \tag{1}$$

Later, Galileo considers in which ways this insurmountable limit can be increased. He envisaged two ways: or to increase the material scope $\mathcal{A}$, or perhaps to change the shape of the column. In the latter case, he reasons —in a funny paragraph— that if the giants exist, they would have a very different aspect and proportions that human being, specifically the bones of their legs would have a diameter/length ratio greater, because otherwise their weight —that increases proportional to L^3— would be greater in proportion to their strength —that increases as L^2—, and as a result the giants would suffer stresses —that increases as L— greater than human beings, and being the bone material very similar in all the mammals, the giants would be unable to performs in their life as well as human beings. A few centuries later, this result could be confirmed comparing dinosaurs of different size but of same suborder or family (i.e., *Theropoda* or *Tyrannosauridae*) [19].

As a material with infinite strength or null specific weight does not exist, it is clear that following the first way we can only increase the insurmountable height but remaining finite. If we adopt the latter way, the main question arises: does it exist an optimal shape which have infinite height? We are looking for an answer to this question because it is a key into the theory of structural design. If the answer is "Yes", then the Galileo's problem have a solution for any size considered. But if the answer is "No", there exists instances of the problem which have no solution, i.e., there are unsolvable problems in structural design. Furthermore, as we will show below, near the unsurmountable size, any solution for the problem will have an unafardobla physical cost, so it wuold be infeasible from a practical view.

Our working hypothesis is that a finite insurmountable size exists for a fairly large set of structural problems (not only for Galileo's problem). Moreover, the optimal shape for each problem —that maximise the finite insurmountable size— is a sound reference to measure the efficiency of all other shapes with size lesser than the one of the optimal shape [4, 3, 21].

In a first approximation, we can represent the physical cost of a structure by its self-weight, as many cost during the manufacturing, but not all, are approximately proportional to the self-weight of the structure: CO_2 emissions, mineral resources consumption, etc. For a given structural problem, we define the structural efficiency as the ratio between the useful load and the total load (i.e., the useful load plus the self-weight) required to solve that problem in a particular structure. Galileo postulated also the relationship between the size of a structure and its ability to resist a useful load. Let us consider a cylindrical column of size $L < \mathcal{L}$. It can resist an additional useful load Q, the value of which is at most the weight difference between this column and the column of insurmountable height $\mathcal{L}$. Hence, according to the previous definition of efficiency:

$$r = \frac{\mathcal{L} - L}{\mathcal{L}} = 1 - \frac{L}{\mathcal{L}} \tag{2}$$

Note that (2), that we name Galileo's rule, is exact in the case of cylindrical columns, but it is not proved that it would be a general rule. The best result we get up to date is that Galileo's rule is a very good estimate in canonical problems, like bending of beams and bridges [21]. We define the load cost C as the inverse of efficiency, hence always higher than unity, $C = 1/r$. Then, the self-weight of the column is:

$$P = (C - 1)Q \tag{3}$$

As a reward, Galileo's rule, apart of the cost, gives us an sound estimate for the self-load, that it is a required datum for the final project but unknown in the preliminary phases of design of large structures.

2.1 Comparison between Euler's and Galileo's problems

Remind the cylindrical column of height L and diameter d. As we saw, such a column will be unsafe if its height L be equal to or greater than material scope $\mathcal{A}$. But the column may also fail by elastic buckling. According to Landau and Lifshitz' Course [17], the critical height for buckling is related to the diameter by:

$$L_{\text{cr}} = 0.792\sqrt[3]{\frac{\mathbf{E}}{\boldsymbol{\rho}}} \cdot d^{2/3} \tag{4}$$

The ratio $\mathbf{E} \div \boldsymbol{\rho}$ is another characteristic length of the structural material. Whereas the scope $\mathcal{A}$ is its specific strength, $\mathcal{E} = \mathbf{E} \div \boldsymbol{\rho}$ is its specific stiffness. Let us define the geometrical slenderness of the column as the ratio $\lambda = L \div d$. Then:

$$\lambda_{\text{cr}} = 0.792\sqrt[3]{\frac{\mathcal{E}}{d}} \qquad \lambda_{\text{cr}}\sqrt[3]{d} = 0.792\sqrt[3]{\mathcal{E}} \tag{5}$$

Therefore, the safety of a given column bearing only its own weight requires that two conditions hold: (i) $L \le \mathcal{A}$; (ii) $\lambda \le \lambda_{\text{cr}}$. It is worth to note that it is always possible to satisfy the second condition, as for each height we can choose d such that $\lambda = \lambda_{\text{cr}}$. However the first condition is an absolute one, as it only depends of the properties of material. Hence, the height of a safe, cylindrical column would be lesser than or equal to $\mathcal{L}$.

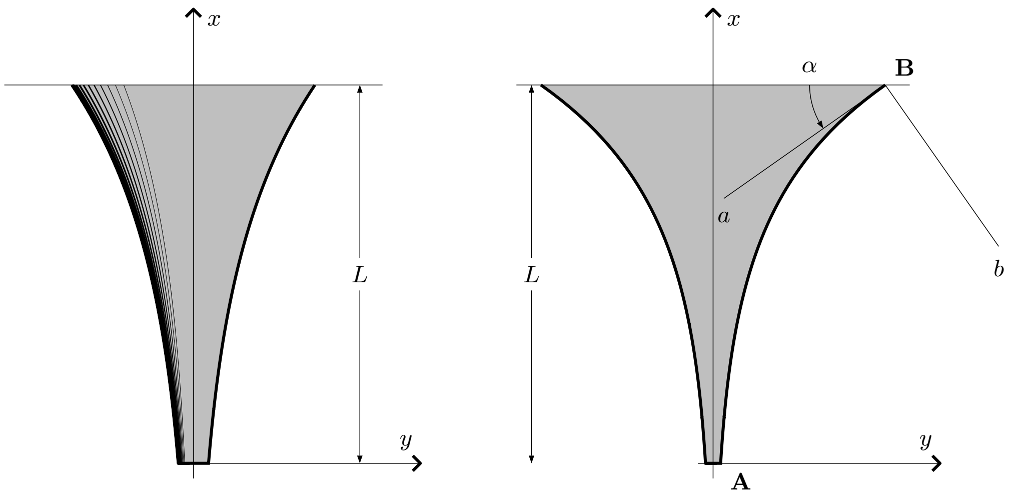
This limit, as noted above, only could be modified in two ways: changing material's properties or changing the shape of the column (or both). The interesting point here is that to answer Galileo's question we must elucidate if a finite structural scope $\mathcal{L}$ (related with the material scope $\mathcal{A}$) exists for any shape of the column. The interesting advances in the analysis of the solution for Euler's problem [7, 8, 10] are useless to this aim, mainly because of ignoring the limit that strength issues imposes on the shape (condition (i) for cylindrical columns).

Furthermore, as it is well-known, the classical solutions of Euler's problems on buckling are contradictory with the experimental data. "This contradiction between theory and experiments is not surprising. The ideal appearance of a phenomenon is always more or less influenced in practise by multiple causes that can deform it to the point of leaving none but a caricature. In the problems of instability, the theory considers only perfect elements, both form and structure indefinitely elastic and resistant. The test pieces, as elements actually built, are very far from perfection: the materials are inhomogeneous, and they are approximately elastic, and within certain limits." [9] As a consequence, in the engineering practise, there is no bifurcation between two different equilibria, on the contrary as the slenderness λ approaches to its critical value λ_{cr}, the failure changes continuously from simple flattening to bending with net compression. In fact, the so named critical load (or Euler's load) is not a "load" at all, but a stiffness of the column against lateral displacement, and the failure occurs for real loads numerically lesser than this stiffness. (Unfortunately, this stiffness can be expressed in load units (N), but it is better understood with stiffness unit (mN $\div$ m), showing that it is a ratio between the bending moment and the lateral displacement.) In fact, although the buckling of real structural members (with negligible self-weight) is a non-lineal problem in a first, mathematical view, therefore candidate to a numerical solution, it is possible to overcome the difficulties and to solve it by a direct albeit non exact formulation [5].

3 On the existence of a finite height for Galileo's tallest column

The epistemological situation of Galileo's problem is analogous to the situation that algorithm designers are when confronting the well-known Theory of NP-Completeness [13]. We, the structural designer, do not know if a finite insurmontable size exists for the problem at hand, hence we can not know in advance if our problem is solvable or not. But if we believe that this limit exists, we can managed at least an aproximate of its value, and armed with this knowledge to take a decision about the solvability of the problem. Indeed, if we know the limit that different types of structures can reach for our problem, we can evaluate aprox the relative merit of each type and select the most promising one. So the existence (or not) of a finite height for Galileo's tallest column is a key point for our everyday work.

Let us consider the two main approaches to the problem: first, that such a limit does not exist because it is easy to find the corresponding shape; second, that such a limit probably exist because it is very hard to find out any shape that can overcome a given finite limit on its height.



(a) 3D-solution (b) 2D-solution

Figure 1: Constant Maximum Strength Cable: classical solution

3.1 The known solutions with infinite height are unfeasible

Although the Euler's tallest column and related problems have received very much attention up to date, some researchs while studying this problem spent a minutes to study too problems related with that of Galileo. This is the case of Karihaloo and Hemp [16], that study the "maximum strength design" of structural members. In their approach, all cross sections of a structural member attain the maximum allowable stress for the given material, therefore the solution is also referred to as "constant maximum strength design". Let us examine with some detail the constant maximum strength design for tension or compression members (section 2.1 of the cited paper), see FIGURE 1.

Consider a cable of length L and cross-sectional area $A(x)$. If the gravity and the external load are acting in the same direction, the condition of the constant maximum strength is:

$$\mathsf{f}A = \int_0^L \boldsymbol{\rho} A \,\mathrm{d}x\,; \qquad \frac{\mathrm{d}A}{A} = \frac{\mathrm{d}x}{\mathcal{A}} \tag{6}$$

The solution is:

$$A = A_0 \cdot \exp(x/\mathcal{A}) \tag{7}$$

If the cross-section is circular, the radius $r(x)$ give us the contour of the member:

$$r(x) = r_0 \cdot \sqrt{\exp(x/\mathcal{A})} = r_0 \cdot \exp\left(\frac{x}{2\mathcal{A}}\right) \tag{8}$$

This solution can have an infinite height with constant stress and bear a useful load ($Q = A_0\mathsf{f}$). Any how, its volume growth exponentially with its size ($A_0\mathcal{A} \cdot (\exp(L/\mathcal{A}) - 1)$), so in practise it is an "intractable" solution —in the same meaning that it is used in algorithm complexity theory [13]—, with a load cost:

$$C = \frac{Q + \rho V(L)}{Q} = 1 + \frac{1}{\boldsymbol{\rho}}\{\exp(L/\mathcal{A}) - 1\} \tag{9}$$

Furthermore, the solution is no feasible from the point of view of equilibrium because only the equilibrium in the direction x is considered for obtaining a constant stress σ_x. Let us consider the 2D-case for the sake of simplicity, see FIGURE 1(B). As the border **AB** is stress-free, the tangent in any point is a principal direction (σ_a), as it is the orthogonal direction ($\sigma_b = 0$). As $\sigma_x = \mathsf{f}$, we have:

$$\begin{aligned} \sigma_a &= \frac{\mathsf{f} + \sigma_y}{2} + \sqrt{\left(\frac{\mathsf{f} + \sigma_y}{2}\right)^2 + \tau^2} \\ 0 &= \frac{\mathsf{f} + \sigma_y}{2} - \sqrt{\left(\frac{\mathsf{f} + \sigma_y}{2}\right)^2 + \tau^2} \end{aligned}$$

and hence $\sigma_a = \mathsf{f}/\cos^2(\alpha)$ if α is the angle between the principal stress direction and coordinate axes. This value is greater than allowable stress f. In fact, as L increases, $\cos\alpha \to 0$ exponentially, and σ_a grows in the same way. As a consequence, the classical solution is not a feasible one for common failure criteria,

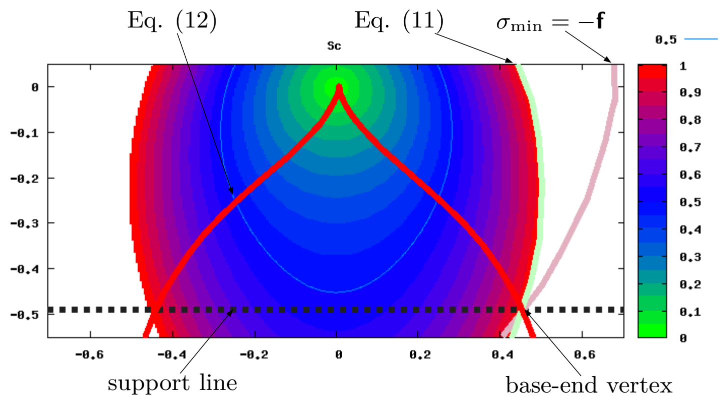


Figure 2: A feasible Galileo's column from a given stress field

i.e., as Von Mises criterion. This error is common to all solutions obtained making use of Bernoulli-Euler theory or Navier hypothesis. These solutions do not proof in any way that an infinite size for columns or beams would be feasible. Of course, these solutions are almost exact when the size L is much lesser than the material structural scope $\mathcal{A}$, as the exponential function grows very slowly when its argument is very small, and for small size the solutions obtained pass fairly well through experimental checks [23]. But they are useless to answer the question pointed out by Galileo, because to this issue we must explore sizes of the same order of magnitude than material scope, i.e., large structures for which the equilibrium equations must be completely fulfilled.

3.2 Trying to refute the existence of a finite height

In 2010, we tried to refute the existence of a finite structural scope whatever be the structural material of the Galileo's column (unpublished work). Our try was naive and unsuccessful. However we think it can help others to understand the difficulties of the problem and perhaps give clues to better searches for a complete solution. To be short, let us consider a 2D-version.

We have an elastic linear material as before. We choose Von Mises criterion as the failure one, hence in any point of a feasible body the following expression holds:

$$\sigma_c = \sqrt{\sigma_x^2 + \sigma_y^2 + 3\tau^2} \leq \mathbf{f} \tag{10}$$

Let us suppose that we have able to determine a suitable stress field for an instance of Galileo's problem, defined over all the xy plane. This field implicitly defined the shape of a body for which (10) holds. The support line must intersect in any point some solution of:

$$\sigma_c(x, y) = \mathbf{f} \tag{11}$$

Let be a and b the principal planes of the stress tensor. Then the curves that can form the stress-free contour of the column will be solutions of one of the following equations:

$$\sigma_a(x, y) = 0 \qquad \sigma_b(x, y) = 0 \tag{12}$$

depending if we are looking for a compressive or tensile solution. Furthermore, these curves intersect in some point the solution of (11) and also the solution of:

$$\sigma_a(x, y) = \pm\mathbf{f} \qquad \sigma_b(x, y) = \pm\mathbf{f} \tag{13}$$

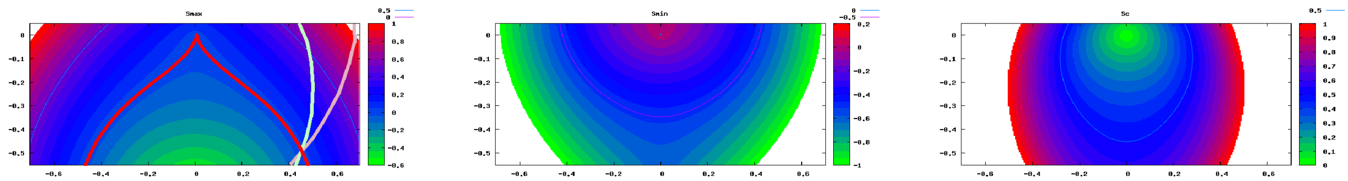
Indeed, if in (12) we select the direction a, we must select now b in (13) and vice versa. This intersection point will be named "base-end vertex" hereafter. Selecting an appropriate set of arcs from solutions of (12), with the additional condition that they define a closed region on the plane together with the support line, we get the shape of a solution for this instance of Galileo's problem and we can determine its height relative to the material scope, v. FIGURE 2.

If we would be able to explore completely the set of all possible stress field —and of course, we are not— with the maximal height obtained we could answer ("Yes" or "No") to Galileo's question.

3.2.1 Generating subsets of stress fields

To check the possibility of our idea we recall on well-known Airy's function. Whatever Airy's stress function Φ, for which the biharmonic equation holds, can be defined as:

$$\Phi = \mathrm{Re}\left[\bar{Z}\Psi(Z) + \mathrm{X}(Z)\right] \tag{14}$$



Left: $\sigma_a = \sigma_{\max}$. Centre: $\sigma_b = \sigma_{\min}$. Right: Von Mises stress σ_c.
The origin of coordinates is up in the centre of the figures, and there the stress tensor is null.
In the left figure, the three curves that define the body —from the stress field and satisfying the Von Mises criterion— are drawn. The red curve is $\sigma_{\max} = 0$, i.e., the stress-free surface of the body. The pale blue curve is $\sigma_c = \mathbf{f}$: the body can not extend beyond. The pale rose curve is $\sigma_{\min} = -\mathbf{f}$: the support line can not extend beyond.
The three curves intersect in a unique point. The support can be the horizontal line for this point: in this way, it can be assured that the stress at the support does not exceed the allowable stress.
The insurmountable size of the body is the absolute value of the ordinate of the intersection point, $\mathcal{A}/2$ in this example.

Figure 3: Stress fields and the corresponding body

where Ψ and X are analytical functions in $\mathbb{C}$ [14]. The function Φ satisfies both compatibility and equilibrium equations, and the displacements can be calculated without integration of the stress functions, from the complex potentials Ψ y X.

The stress field is:

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} - \boldsymbol{\rho} y \qquad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2} - \boldsymbol{\rho} y \qquad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \tag{15}$$

Using the Hooke law with this field, the displacement functions are:

$$\begin{aligned} u(x,y) &= \frac{1}{\mathbf{E}} \left\{ -(1+\nu)\frac{\partial \Phi}{\partial x} + 4 \cdot \mathrm{Re}\left[\Psi(x,y)\right] - \rho(1-\nu)xy \right\} + \theta_0 y + u_0 \\ v(x,y) &= \frac{1}{\mathbf{E}} \left\{ -(1+\nu)\frac{\partial \Phi}{\partial y} + 4 \cdot \mathrm{Im}\left[\Psi(x,y)\right] + \frac{\rho}{2}(1-\nu)\left(x^2 - y^2\right) \right\} - \theta_0 x + v_0 \end{aligned} \tag{16}$$

where ν is Poisson's modulus.

The complex potentials considered were simple polynomials, v. Table 1. The coordinate origin will be at the top of a column (or at the bottom of a cable). Some boundary conditions must hold always:

$$\begin{array}{llll} \text{Symmetry:} & \sigma_x(x,y) = \sigma_x(-x,y) & \sigma_y(x,y) = \sigma_y(-x,y) & \tau_{xy}(x,y) = -\tau_{xy}(-x,y) \\ \text{Origin:} & \sigma_x(0,0) = \sigma_y(0,0) = \tau_{xy}(0,0) = 0 & & \end{array} \tag{17}$$

We considered also different supports:

$$\begin{array}{ll} \text{Base Line support:} & v(x,L) = 0 \\ \text{Base Point support:} & v(0,L) = 0 \end{array} \tag{18}$$

With a concrete selection of boundary conditions and support geometry, an instance of the problem is defined. We considered several of them, v. Table 2.

The trial stress field is derived from Ψ y X, which are defined through some constant parameters to be determined. In this way $\Phi(x,y)$ is completely defined. Let us name $\mathbf{p}$ to the set of parameters to be determined for the expressions of Ψ, X, $u(x,y)$ and $v(x,y)$. This parameters, once determined, will define the body (or bodies) generated by the trial stress field.

Table 1: Complex, polynomial potentials

$$\begin{aligned} \Psi_0 &= A_p + i \cdot B_p \\ \Psi_1 &= (A_p + i \cdot B_p) + (C_p + i \cdot D_p) \cdot Z \\ \Psi_2 &= (A_p + i \cdot B_p) + (C_p + i \cdot D_p) \cdot Z + (E_p + i \cdot F_p) \cdot Z^2 \\ \ldots &= \ldots \\ \Psi^n &= (P_{n0} + iP_{n1}) \cdot Z^n \\ \mathrm{X}_0 &= A_c + i \cdot B_c \\ \mathrm{X}_1 &= (A_c + i \cdot B_c) + (C_c + i \cdot D_c) \cdot Z \\ \mathrm{X}_2 &= (A_c + i \cdot B_c) + (C_c + i \cdot D_c) \cdot Z + (E_c + i \cdot F_c) \cdot Z^2 \\ \ldots &= \ldots \\ \mathrm{X}^n &= (C_{n0} + iC_{n1}) \cdot Z^n \end{aligned}$$

3.2.2 Getting a feasible body

Once a trial stress field is selected in algebraic form, the Eqs. (17) and (18) must hold simultaneously for all x, y. Let us represent this condition with the set of problem equations:

$$\mathbf{P} = \mathbf{0} \tag{22}$$

It is worth to note that into (22), all the equations needed for satisfying (17) or (18) for all x, y must be included. For example, if one of (18) would be:

$$(A^2 - CB)x^2 + (L - DA)xy + (E^3 - ABC)y^2 = 0$$

then the following three equations would be included into (22):

$$A^2 - CB = 0 \qquad L - DA = 0 \qquad E^3 - ABC = 0$$

Solving (22) for all x, y, we obtain a set of relationships between the parameters of $\mathbf{p}$, that we can wrote as

$$\mathbf{p} = \mathbf{Q}\mathbf{q} \tag{23}$$

where $\mathbf{q}$ is the set of independent parameters for a given $\Phi(x, y, \mathbf{q})$. It is worth to note that some (or all) parameters in $\mathbf{q}$ can disappear from the stress or displacement fields, because only derivatives of Φ are present in the expressions of these fields.

If the stress field depend of some components of $\mathbf{q}$, these components can be chosen freely as equilibrium or compatibility equations will hold for arbitrary values of $\mathbf{q}$. This fact means that the given Φ represent a family or set of solutions, not a unique one.

Remind that σ_c is the comparison stress, and it must be calculated to assert if the solution is feasible or not, i.e., $\sigma_c \leq \mathsf{f}$. The given or unknown line of the support (straight or curve) must to intersect to some solutions of:

$$\sigma_c(x, y, \mathbf{q}) = \mathsf{f} \tag{24}$$

This requirement is necessary if the body attains its maximal resistance.

Let be a and b the principal planes of the stress tensor. Then the curves that can form the contour free of the stress would be solutions of one of the following equations:

$$\sigma_a(x, y, \mathbf{q}) = 0 \qquad \sigma_b(x, y, \mathbf{q}) = 0 \tag{25}$$

We must choose one of the two, depending if we look for a column (compression) or a cable (tension). The solutions would be curves of the form $F(x, y, \mathbf{q}) = 0$.

Selecting support curves with free contour curves in such a manner that they form a convex domain, a set of shapes is determined. For each shape, the safety criterion must be imposed in any interior point. In this phase, some parameters of $\mathbf{q}$ could be dependent of others, and in this case a new reduction of the number of parameters results:

$$\mathbf{q} = \mathbf{R}\mathbf{r}$$

If all the parameters become determined now, the stress field correspond to an unique shape. Otherwise, we have a family of shapes.

The last problem is to determine the base-end vertex mentioned before. If the curves that define the shape can be obtained in explicit form ($y = f(x)$), they can be managed directly. Otherwise, the shape will be defined by inequations, whose sign can be determined for each function considering the sign of its value in $(0, L/2)$, where L is the size used in (18), if the origin was specified as a point of the free contour (null surface stress) in (17).

Anyhow, always it is possible simply to draw the curves of the contour of the body: $\sigma_a = 0$ and the support line for an arbitrarily chosen value of the size L. The last one will intersect to the former and to $\sigma_b = \pm\mathsf{f}$ in two different points. But these two points may be the same base-end vertex of the contour: the first point mentioned is the vertex of the base where the stress-free contour ends; the second one is the vertex where the own base ends. Hence, adjusting the value of L in such a manner that the two points be the same, we will find out the correct base-end vertex and the height of the shape.

3.2.3 An example in detail

Definitions

- Complex Potentials: $\Psi = \Psi^5 + \Psi^6 \quad \mathrm{X} = \mathrm{X}^3 + \mathrm{X}^4$ v. TABLE 1
- Problem and support: `MountainBis` (TABLE 2)
- $\mathbf{p}^{\mathrm{T}} = \{\theta_0, u_0, v_0, C_{30}, C_{31}, C_{40}, C_{41}, P_{50}, P_{51}, P_{60}, P_{61}\}$

TABLE 2: STANDARD PROBLEM DEFINITIONS

`Mountain or Peak` A shortening of the shape can be measured with $v(0,0) > 0$.

$$\begin{array}{l} \sigma_x(x,y) = \sigma_x(-x,y) \quad \sigma_y(x,y) = \sigma_y(-x,y) \quad \tau_{xy}(x,y) = -\tau_{xy}(-x,y) \\ \sigma_x(0,0) = \sigma_y(0,0) = 0 \\ \text{Support line: } y = H \\ v(x,H) = 0 \quad u(0,y) = 0 \quad u(0,H) = 0 \end{array} \tag{19}$$

`MountainBis` A shortening of the shape can be measured with $v(0,H) > 0$.

$$\begin{array}{l} \sigma_x(x,y) = \sigma_x(-x,y) \quad \sigma_y(x,y) = \sigma_y(-x,y) \quad \tau_{xy}(x,y) = -\tau_{xy}(-x,y) \\ \sigma_x(0,0) = \sigma_y(0,0) = 0 \\ \text{Support line: } y = H \\ \partial v(x,H)/\partial x = 0 \quad u(0,0) = 0 \quad u(0,H) = 0 \quad v(0,0) = 0 \end{array} \tag{20}$$

`MountainIII or Summit` A shortening of the shape can be measured with $v(0,H) > 0$.

$$\begin{array}{l} \sigma_x(x,y) = \sigma_x(-x,y) \quad \sigma_y(x,y) = \sigma_y(-x,y) \quad \tau_{xy}(x,y) = -\tau_{xy}(-x,y) \\ \sigma_y(0,0) = 0 \\ \text{Support: } y = H \\ \partial v(x,H)/\partial x = 0 \quad u(0,0) = 0 \quad u(0,H) = 0 \quad v(0,0) = 0 \end{array} \tag{21}$$

TABLE 3: SHAPES OF 'MOUNTAIN' FOUND

[normal steel, dimensionless properties: $\nu = 0.25$; $\rho = \mathbf{f} = 1$; $\mathbf{E} = 808$]

problem	Potentials	$\mathcal{L} \div \mathcal{A}$	base $\div\, \mathcal{A}$	shape
`Mountain`	$\{\Psi_2; \mathrm{X}_1\}$	$0,389$	$6,96$	triangle
`Mountain`	$\{\Psi^4; \mathrm{X}^4\}$	$0,50$	$1,53$	ellipse segment
`MountainBis`	$\{\Psi^5 + \Psi^6; \mathrm{X}^3 + \mathrm{X}^4\}$	$0,5$	$f(C_{31})$	*pseudo*exponential
`Peak`	$\{\Psi_3; \mathrm{X} = 0\}$	$\frac{8}{11}$	$f(P_{21})$?	parabolic segment base $\frac{32\sqrt{2}}{11\sqrt{3}}$ for $P_{21} = 0$

- $\mathbf{q}^{\mathrm{T}} = \{C_{31}\}$
- Stress function:

$$\Phi(x,y) = -\frac{\left(y^4 - 6\,x^2\,y^2 + x^4\right)\,\left((6\,\nu+6)\,\,C_{31} + (\nu-1)\,\,\rho\right)}{(24\,\nu+24)\,\,H} - \left(3\,x^2\,y - y^3\right)\,\,C_{31}$$

- Stress tensor:

$$\sigma_x(x,y) = \frac{(12\,(\nu+1)\,\,y\,C_{31} - 2\,(\nu-1)\,\,\rho\,y)\,\,H + 6\,(\nu+1)\,\left(x^2-y^2\right)\,C_{31} + (1-\nu)\,\,\rho\,\,\left(y^2-x^2\right)}{2\,(\nu+1)\,\,H}$$

$$\sigma_y(x,y) = -\frac{2\,(\nu+1)\,\,y\,(6C_{31}+\rho\,)\,H + 6\,(\nu+1)\,\,\left(x^2-y^2\right)\,C_{31} + (1-\nu)\,\,\rho\,(y^2-x^2)}{2\,(\nu+1)\,\,H}$$

$$\tau_{xy}(x,y) = \frac{6\,(\nu+)\,\,x\,C_{31}(H-y) + (1-\nu)\,\,\rho\,x\,y}{(\nu+1)\,\,H}$$

- Displacements:

$$u(x,y) = \frac{6\,(6\,(\nu+1)\,\,x\,y\,C_{31} + (\nu-)\,\,\rho\,x\,y)\,\,H + 6\,\left(-3\,(\nu-1)\,\,x\,y^2 + (\nu+1)\,\,x^3\right)\,C_{31} + (1-\nu)\,\,\rho\,(3\,x\,y^2 - \,x^3)}{6\,\mathbf{E}\,H}$$

$$v(x,y) = -\frac{6\,(3\,(\nu+1)\,\,C_{31} - (1-\nu)\,\,\rho)\,(y^2-x^2)\,H + 6\,(\nu+1)\,(3\,x^2\,y - y^3)\,C_{31} + (1-\nu)\,\,\rho\,(y^3 - 3\,x^2\,y)}{6\,\mathbf{E}\,H}$$

Bodies There is a family dependent on parameter C_{31} de X. This parameter defines the abscissa of the base-end vertex.

- Vertices: $V_1 = (f(C_{31}), H)$; $V_2 = (0,0)$; $V_3 = (-f(C_{31}),, H)$. For a normal steel (v. Table 3), the real domain of f is approx $(-0.43; 0.23)$ with roots in the extremes of the interval and in 0.1:

$$f_{\mathrm{std}}(C_{31}) = \frac{\sqrt{10\sqrt{136C_{31}^2 - 20C_{31} + 1} - 300C_{31}^2 - 3}}{10\sqrt{3}C_{31} - \sqrt{3}}$$

- Stress-free arcs. They have not analytical expressions. Drawing them it is clear that there are solutions only for $C_{31} \in [-0.165; 0.1]$.

  The base width varies from 0,9 up to 1,4$\mathcal{A}$. The height of all shapes is constant, only depending on material properties, , v. Figure 3. .

- Insurmountable size $\mathcal{L}$: 0.5$\mathcal{A}$

3.2.4 Provisional conclusion

After our research, we can claim nothing about Galileo's question. But at least we have tried to do the best: trying to refute our own conjectures. Maybe it is possible to search on all the set of complex potentials with methods of high mathematics, which of course are beyond our knowledge.

4 Formal definitions of Galileo's problem

After all, we have two main hypothesis that we outline informally as follows:

Conjecture 1 *A finite insurmountable size exists for a fairly large set of structural problems (not only for Galileo's problem) when the self-weight and stress limit are taken in consideration.*

The second one is suggested for the results of our search in §3.2, and it is strongest that the first one:

Conjecture 2 *In the case of original Galileo's problem, the insurmountable size for a solid column (without holes of any kind) is equal to the material scope, i.e.,* $\mathcal{L} = \mathcal{A}$

To refute any of both hypothesis consists in showing a given problem —including support, boundary conditions, and failure criterion on stresses— and a shape family for structures that can solve it, that includes a shape of infinite size.

4.1 The 2D Galileo's problem

For the sake of simplicity let us consider a 2D-universe.

Problem 1 (Galileo, 1638) *To find a y-symmetrical body of maximal height, placed in the semi-plane $y > 0$ and supported in the $y = 0$ line, only bearing its own weight, of an homogeneous, linear elastic material defined by Young's Modulus E, Poisson's modulus ν, allowable compressive stress f and specific weight $\boldsymbol{\rho}$, and subjected to displacement constraint $v(x, 0) = 0$, $u(0, y) = 0$, and to some criterion $\mathcal{C}$ on stresses, to be fulfilled over all the body, that it can be expressed as:*

$$\mathcal{C}: \quad \sigma_{\mathcal{C}}(x, y) \leq \mathsf{f} \quad \forall (x, y) \in \textit{body} \tag{26}$$

The support line can be a surface with friction following Coulomb theory. Hence the tangential stress is subjected to:

$$\mathrm{abs}\left(\tau(x, 0)\right) \leq \mu\sigma_y \quad \textit{with } \mu > 0 \tag{27}$$

There is not additional fundamental constraints on shape, but someone can be imposed for convenience.

The original Galileo column is simply a rectangular domain of height $\mathcal{A}$ and whatever width w. The principal stresses are $\sigma_b = \sigma_y = -\mathsf{f}(1 - y \div \mathcal{A})$ and $\sigma_a = \sigma_x = 0$. Further, the Von Mises stress is $\sigma_{VM} = \mathsf{f}(1 - y \div \mathcal{A})$.

If any one can envisaged a general proof of our hypothesis, the related problems would be directly solved. In any case else, the general formulation of the problem can be stated as to find out a shape with infinite size or height, proving in this way that our working hypothesis is false but being the shape of maximum scope determined. We think on this class of problems as good candidates for some topology or shape optimisation methods [2, 20, 22, 25, 26].

4.2 Other related problems

As the general formulation can be hard to attack with available methods, we can suggest some alternative problems which in our view could be equivalent (or at least approximately equivalent) to the original.

Let be $V_0 = \mathcal{A}w$ a given volume in the 2D-universe. We can consider the problem of finding a shape with this given volume that maximise the height of the figure subject to the same stress tensor constraint.

Perhaps the stress constraint can be replaced by minimising the (maximum or mean) Von Mises stress in the volume, being the latter unbounded, and the total height of the figure fixed to a given value $\mathcal{A}$. With this problem it should be the case that we will get solutions with maximum absolute Von Mises stress lesser than f, hence with appropriated scaling we will get a solution higher than Galileo's column.

Another approach arises from considering the calculus of the maximum scope of a shape as a limit case. Let us consider an useful load at a height $y = L > 0$ as an uniform load p along a width w_0. The problem is now to find a shape of minimal weight in equilibrium with p and its self-weight with the stress constraints as above. One additional constraint on the shape will be that it must lie into the region limited by $y \leq L$ and $y \geq 0$. If this problem can be solved, the structure scope $\mathcal{L}$ will be the limit of L when $p \to 0$ or $w_0 \to 0$. Obviously, a solution is Galileo's column of constant width equal to w_0, but is there another one? The useful load can be defined too as $P = \int_{-w_0/2}^{w_0/2} p(x)\,\mathrm{d}x$ being P a given constant. In this case the function p can be viewed as a design variable, or its integral over w_0 as a additional constraint on stresses.

More equivalents formulations can exist or can be proposed following these lines.

We think that a minimum compliance approach it is not equivalent to problems in Galileo's realm due to self-weight. But it could be the case that minimum compliance objective leads to useful solutions that after appropriate scaling provide that the stress constraint be fulfilled.

5 Conclusion

Galileo's problem has theoretical interest for mathematics and very practical interest for the structural design theory. It would be a benchmark problem for topology or shape optimisation methods. Each different stress constraint or material model (e.g., plasticity) lead to new instances of the problem. By 2038 it will make four centuries that it was formulated.

We will appreciate any insight on it.

References

[1] Joaquín Antuña and Mariano Vázquez Espí. ¿Existen problemas estructurales irresolubles? Una cuestión abierta. *Informes de la Construcción*, 64(525), 2012.

[2] Hideyuki Azegami, Shota Fukumoto, and Taiki Aoyama. Shape optimization of continua using NURBS as basis functions. *Struct Multidisc Optim*, 45?, 2012.

[3] J. Cervera and M. Vázquez. Galileo, Maxwell, Michell, Aroca: measuring the structural efficiency. In *Structural Milestone in Architecture and Engineering. International Conference on Research in Construction*, Madrid, november 2011. IETcc–UPM, Instituto de Ciencias de la Construcción (CSIC). (Also available at `http://oa.upm.es/9931/`).

[4] Jaime Cervera Bravo. Las estructuras y el peso propio. *Informes de la Construcción*, 42(407):73–85, Junio 1990.

[5] Jaime Cervera Bravo, Jesús Ortiz Herrera, Mariano Vázquez Espí, and Antonio Aznar López. Dimensionado en compresión en acero: el peso del pandeo. *Rev. Int. Mét. Num. Cálc. Dis. Ing.*, 29(1):79–91, 2013. (In press.

[6] Y.H. Chai and C.M. Wang. Approximate Solution for the Shape of Submerged Funicular Arches with Self-weight. *Journal of Structural Engineering*, 131(3):399–404, 2005.

[7] Steven J Cox and C McCarthy. The shape of the tallest column. *SIAM Journal on Mathematical Analysis*, 29(3):547–554, 1998.

[8] JI Díaz and M Sauvageot. Euler's tallest column revisited. *Nonlinear Analysis: Real World Applications*, 11(4):2731–2747, 2010.

[9] J Dutheil. Theorie de l'instabilite par divergence d'equilibre. In *4th congress IABSE*, pages 275–295. Association Internationale des Pont et Charpentes, 1952.

[10] Youri V Egorov. On the tallest column. *Comptes Rendus Mécanique*, 338(5):266–270, 2010.

[11] Leonhard Euler. Sur la force des colonnes. *Memoires de L'Academie des Sciences et Belles-Lettres*, 13:252–282, 1759.

[12] Galileo Galilei. *Discorsi e Dimostrazioni Matematiche*. Elsevierii, Leiden, 1638.

[13] Michael R Garey and David S Johnson. *Computers and intractability: a guide to NP-completeness*. WH Freeman New York, 1979.

[14] Édouard Goursat. Sur quelques équations de monge intégrables explicitement. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 1(1-2):35–59, 1932.

[15] John BS Haldane. On being the right size. *Harper's Magazine*, 152:424–427, 1926.

[16] B.L. Karihaloo and W.S. Hemp. Maximum stregth/stiffness design of structural members in presence of self-weight. *Proc. R. Soc. Lond. A*, 389:119–132, 1983.

[17] LD Landau and EM Lifshitz. *Course of Theoretical Physics. Theory of Elasticity*, volume 7. Pergamon Press, 2nd english ed edition, 1970.

[18] TA McMahon. Shape and size in biology. *Science*, 179:1202–1204, 1973.

[19] Thomas A McMahon, John Tyler Bonner, and WH Freeman. *On size and life*. Scientific American Library New York, 1983.

[20] F. Navarrina, I. Muiños, I. Colominas, and M. Casteleiro. Topology optimization of structures: a minimum weight approach with stress constraints. *Advances in Engineering Software*, s.d.:s.d., 2005.

[21] Carlos Olmedo Rojas, Mariano Vázquez Espí, and Jaime Cervera Bravo. On the insurmountable size of truss-like structures. In *Proceedings of Third International Conference on Mechanical Models in Structural Engineering*. University of Seville, 2015. (submitted).

[22] J. París, F. Navarrina, I. Colominas, and M. Casteleiro. Stress constraints sensitivity analysis in structural topology optimization. *Computer Methods in Applied Mechanics and Engineering*, 199:2110–2122, 2010.

[23] Raymond H Plaut and Lawrence N Virgin. Optimal design of cantilevered elastica for minimum tip deflection under self-weight. *Structural and Multidisciplinary Optimization*, 43(5):657–664, 2011.

[24] D'Arcy Wentworth Thompson. *On growth and form*. Cambridge Univ. Press, 1942.

[25] Mariano Victoria, Pascual Martí, and Osvaldo M. Querin. Topology design of two-dimensional continuum structures using isolines. *Computers and Structures*, 87:101–109, 2009.

[26] Bin Zheng, Ching jui Chang, and Hae Chang Gea. Topology optimization considering body forces. *International Journal for Simulation and Multidisciplinary Design Optimization*, 3:316–320, 2009.